ELSEVIER

Topology and its Applications 85 (1998) 281–285

TOPOLOGY AND ITS APPLICATIONS

# An example of $t_p^*$-equivalent spaces which are not $t_p$-equivalent

Witold Marciszewski [a,b,1], Jan van Mill [a,*]

[a] Vrije Universiteit, Faculty of Mathematics and Computer Science, De Boelelaan 1081a, 1081 HV Amsterdam, The Netherlands

[b] University of Warsaw, Warsaw, Poland

Received 15 October 1996

**Abstract**

We construct an example of two countable spaces $X$ and $Y$ such that the spaces $C_p^*(X)$ and $C_p^*(Y)$ are homeomorphic and the spaces $C_p(X)$ and $C_p(Y)$ are not homeomorphic. © 1998 Elsevier Science B.V.

*Keywords:* Function space; $C_p(X)$; Ultrafilter

*AMS classification:* 54C35

## 1. Introduction

All spaces are completely regular.

For a space $X$, $C_p(X)$ denotes the space of all continuous real valued functions on $X$ with the pointwise convergence topology. $C_p^*(X)$ is the subspace of $C_p(X)$ consisting of bounded functions.

Recently, Banakh and Cauty [2] proved that if $X$ is countable and nondiscrete then $C_p^*(X)$ is homeomorphic to $C_p(X) \times \sigma$, where $\sigma$ denotes the linear span of the standard basis in $\ell^2$. This interesting result has several nontrivial consequences, among them the statement that if $C_p(X)$ and $C_p(Y)$ are homeomorphic then so are $C_p^*(X)$ and $C_p^*(Y)$. This result suggests the natural question of whether the reverse implication holds. The aim of this note is to answer this question in the negative: there exist countable spaces $X$

* Corresponding author. E-mail address: vanmill@cs.vu.nl.

1 E-mail address: wmarcisz@cs.vu.nl.



PII S0166-8641(97)00154-5

and $Y$ for which $C_p^*(X) \approx C_p^*(Y)$ (i.e., $X$ and $Y$ are $t_p^*$-equivalent) but $C_p(X) \not\approx C_p(Y)$ (i.e., $X$ and $Y$ are not $t_p$-equivalent). For a related result, see [1].

Given a filter $F$ on an infinite countable set $T$, we denote by $N_F$ the space $T \cup \{\infty\}$, where $\infty \notin T$, topologized by isolating the points of $T$ and using the family $\{A \cup \{\infty\}\colon A \in F\}$ as a neighborhood base at $\infty$.

Recall that a filter $F$ is a $P$-filter if for every sequence $(U_n)$ of sets from $F$ we can find an $A \in F$ which is almost contained in every $U_n$, i.e., $A \setminus U_n$ is finite. $P$-ultrafilters are also called $P$-points. For the notions from infinite-dimensional topology that we are using, we refer the reader to [9].

## 2. The example

**Lemma 2.1.** *Let $F$ be a filter on $\omega$ which is not a $P$-filter. Then the intersection $C_p(N_F) \cap [0,1]^{N_F}$ contains a closed nonempty subset $R$ which is an absolute retract and a $Z_\sigma$-space.*

**Proof.** Since $F$ is not a $P$-filter we can find a partition of $\omega$ into disjoint infinite subsets $A_k$, $k \in \omega$, with the following properties:

(a) $(\forall i \in \omega)\ [U_i = \bigcup_{k \geqslant i} A_k \in F]$,

(b) $(\forall A \in F)(\exists i \in \omega)\ [A \setminus U_i$ is infinite$]$.

The condition (b) is obviously equivalent to

(b′) $(\forall A \in F)(\exists k \in \omega)\ [A \cap A_k$ is infinite$]$.

To simplify the notation we may assume that $F$ is a filter on $\omega \times \omega$ and $A_k = \{(k,n)\colon n \in \omega\}$.

Let $P$ be the subset of $[0,1]^{N_F}$ consisting of all functions $f$ with the following properties:

(1) $(\forall k,n,i \in \omega)\ [(i \leqslant n) \Rightarrow (f(k,i) \leqslant f(k,n))]$,

(2) $(\forall k,n,i \in \omega)\ [(k \leqslant i) \Rightarrow (f(k,n) \geqslant f(i,n))]$,

(3) $f(\infty) = 0$.

Put $R = P \cap C_p(N_F)$. Obviously, the set $R$ is closed in $C_p(N_F) \cap [0,1]^{N_F}$. Since $R$ is a convex subset of the product $\mathbb{R}^{N_F}$ it is an absolute retract.

Consider an arbitrary $f \in R$. From the continuity of $f$ at $\infty$ it follows that, for every $\varepsilon > 0$, there is an $A \in F$ such that $f(k,n) \leqslant \varepsilon$ for all $(k,n) \in A$. By (b′) there is $k \in \omega$ such that $f(k,n) \leqslant \varepsilon$ for infinitely many $n \in \omega$. Then the condition (1) implies that $f(k,n) \leqslant \varepsilon$ for all $n \in \omega$. From (2) it follows that $f(i,n) \leqslant \varepsilon$ for all $n \in \omega$ and all $i \geqslant k$. Hence we have

$$R = \{f \in P\colon (\forall \varepsilon > 0)(\exists k \in \omega)(\forall i \geqslant k)(\forall n \in \omega)\ [f(i,n) \leqslant \varepsilon]\}.$$

Let

$$R_k = \{f \in R\colon (\forall i \geqslant k)(\forall n \in \omega)\ [f(i,n) \leqslant 1/2]\},$$

for $k \in \omega$. Then each $R_k$ is a closed subset of $R$ and $R = \bigcup\{R_k\colon k \in \omega\}$. One can easily verify that all $R_k$ are $Z$-sets in $R$. Indeed, it is enough to observe that, for a fixed $k$, the sequence of maps $\varphi_j : R \to R \setminus R_k$, $j \in \omega$, defined by

$$\varphi_j(f)(i,n) = \begin{cases} 1 & \text{for } i \leqslant k \text{ and } n \geqslant j, \\ f(i,n) & \text{otherwise} \end{cases}$$

is uniformly convergent to the identity on $R$ (uniformly with respect to any metric on the product $[0,1]^{N_F}$). Therefore $R$ is a $Z_\sigma$-space. □

**Example 2.2.** There exist countable spaces $X$ and $Y$ such that the spaces $C_p^*(X)$ and $C_p^*(Y)$ are homeomorphic and the spaces $C_p(X)$ and $C_p(Y)$ are not homeomorphic.

Let $F$ be an ultrafilter on $\omega$ which is not a $P$-point. We take $X = \omega \times N_F$. Let $S = \{0, 1, 1/2, 1/3, \ldots\}$ (a convergent sequence). The space $Y$ is the topological sum of the spaces $X$ and $S$. We have the following:

**Lemma 2.3.** *The spaces $C_p(X)$ and $C_p(Y)$ are not homeomorphic.*

**Proof.** By [6, Example 7.1], the space $C_p(N_F)$ is a Baire space. Since the space $C_p(X)$ is homeomorphic to $C_p(N_F)^\omega$ it is also a Baire space, see [10].

On the other hand, it is known that the space $C_p(S)$ is homeomorphic to $\sigma^\omega$ (see [4]) and therefore it is of the first category. Hence the space $C_p(Y)$ which is homeomorphic to $C_p(X) \times C_p(S)$ is also of the first category. □

**Lemma 2.4.** *The spaces $C_p^*(X)$ and $C_p^*(Y)$ are homeomorphic.*

**Proof.** Both spaces $C_p^*(X)$ and $C_p^*(Y)$ are $\sigma$-precompact (i.e., they lie in the $\sigma$-compact subsets of their completions). By [2, Corollary 2.7], it is enough to show that each of these spaces embeds as a closed set into the other. The space $C_p^*(Y)$ is homeomorphic to $C_p^*(X) \times C_p^*(S) = C_p^*(X) \times C_p(S) \approx C_p^*(X) \times \sigma^\omega$. Obviously $C_p^*(Y)$ contains a closed copy of $C_p^*(X)$. Since $C_p^*(X)$ is homeomorphic to $C_p^*(X) \times C_p^*(X)$ it remains to prove that $C_p^*(X)$ contains a closed copy of $\sigma^\omega$. From Lemma 1 it follows that the space $T = C_p(N_F) \cap [0,1]^{N_F}$ contains a closed subset $R$ which is an absolute retract and a $Z_\sigma$-space. By Lemma 5.3 from [5], the product $R^\omega$ contains a closed copy of $\sigma^\omega$. Since it is obvious that the product $T^\omega$ can be embedded as a closed subset of $C_p^*(X)$, we are done. □

## 3. Remarks

**Remark 3.1.** Using the results from [3, Lemma 4.11] and [2, Lemma 3.1] it is possible to give slightly simpler examples of spaces $X$ and $Y$ as in Example 2.2. It is enough to consider $X = N_F$, where $F$ is an ultrafilter on $\omega$ which is not a $P$-point, and again take $Y = X \oplus S$. But in this case the proof of the properties of $X$ and $Y$ is much more involved.

During the 8th Prague Topological Symposium, S.P. Gul'ko announced the following result:

**Theorem 3.2** (Gul'ko, Sokolov). *Let $F$ be an ultrafilter on $\omega$. The following are equivalent:*

(i) *$F$ is not a $P$-point,*
(ii) *$C_p(N_F)$ contains a closed copy of the rationals $\mathbb{Q}$ (equivalently $C_p(N_F)$ is not a hereditary Baire space),*
(iii) *$C_p(N_F)$ contains a closed copy of the space $\sigma$.*

Hence we may use this result (together with Lemma 3.1 from [2]) for the proof of the properties of our example, instead of Lemma 2.1. But we decided to include this lemma to make our paper more self-contained.

The result of Gul'ko and Sokolov shows that the existence of $P$-points in $\omega^*$ (which follows from the Continuum Hypothesis) implies the existence of hereditary Baire spaces $C_p(N_F)$. This fact has some other interesting consequences for the function spaces $C_p^*(X)$. We have the following simple observations:

**Proposition 3.3.** *Let $X$ be a countable space such that $C_p(X)$ is a hereditary Baire space. Then the space $C_p^*(X)$ does not contain a closed copy of the space $\sigma^\omega$. In particular $C_p^*(X)$ is not homeomorphic to $C_p^*(X)^\omega$.*

**Proof.** We have $C_p^*(X) = \bigcup_{n=1}^\infty C_p(X) \cap [-n,n]^X$. The Hurewicz theorem implies that, in our case, every closed absolute Borel subset of $C_p(X) \cap [-n,n]^X$ is an absolute $G_\delta$, for every $n$. Therefore every closed absolute Borel subset of $C_p^*(X)$ is an absolute $G_{\delta\sigma}$. It is well known that the space $\sigma^\omega$ is not such a space. The last part of the proposition follows from the fact that $C_p^*(X)$ always contains a closed copy of $\sigma$. (This can be seen by a direct argument, but also follows from the result of Banakh and Cauty quoted in the introduction.) □

Let us point out that the space $C_p^*(N_F)$ is always homeomorphic to all its finite powers $(C_p^*(N_F))^n$. If $X$ is nondiscrete and $C_p^*(X)$ is analytic then it is homeomorphic to $C_p(X)$ and contains a closed copy of $\sigma^\omega$ (see [2, Section 3]). Proposition 3.3 implies the following fact:

**Proposition 3.4.** *Let $F$ be a filter such that $C_p(N_F)$ is a hereditary Baire space. Then the space $C_p^*(\omega \times N_F)$ is not homeomorphic to $(C_p^*(N_F))^\omega$.*

**Proof.** By Lemma 4.11 from [3] the space $C_p(\omega \times N_F) \approx (C_p(N_F))^\omega$ can be embedded as a closed subset of $C_p(N_F)$. Therefore $C_p(\omega \times N_F)$ is a hereditary Baire space and by Proposition 3.3 $C_p^*(\omega \times N_F)$ does not contain a closed copy of the space $\sigma^\omega$. On the other hand $C_p^*(N_F)$ always contains a closed copy of $\sigma$, so $(C_p^*(N_F))^\omega$ contains a closed copy of $\sigma^\omega$. □

Observe that if $X$ is the topological sum of the spaces $X_i$, $i \in I$, then $C_p(X)$ is canonically (linearly) homeomorphic to $\prod_{i \in I} C_p(X_i)$. By [2, Theorem 3.2], if $I = \omega$ and, for all $i \in \omega$, $X_i$ is nondiscrete and $C_p(X_i)$ is analytic, then also

$$C_p^*(X) \approx \prod_{i \in \omega} C_p^*(X_i).$$

Our result shows that this cannot be extended to the general case. □

**Remark 3.5.** In [8, Lemma 4.1] it has been proved that a closed zero-dimensional subset of the space $C_p(N_F)$ can be embedded in $F^\omega$ as a closed subset, for every filter $F$ (here, we consider $F$ as a subspace of the Cantor set $2^\omega \approx \mathcal{P}(\omega)$). The theorem of Gul'ko and Sokolov shows that this is not the case for $C_p^*(N_F)$. This space always contains a closed copy of the rationals $\mathbb{Q}$. But, if $F$ is a $P$-point then the product $F^\omega$ is hereditary Baire. This follows from the fact that $F^\omega$ can be embedded as a closed subset in $C_p(N_F)$, see [3, Lemma 4.11] and [7, Theorem 2.1].

## References

[1] J. Baars, J. de Groot, J. van Mill and J. Pelant, An example of $\ell_p$-equivalent spaces which are not $\ell_p^*$-equivalent, Proc. Amer. Math. Soc. 119 (1993) 963–969.

[2] T. Banakh and R. Cauty, Universalité forte pour les sous-ensambles totalement bornés. Applications aux espaces $C_p(X)$, Preprint.

[3] R. Cauty, T. Dobrowolski and W. Marciszewski, A contribution to the topological classification of the spaces $C_p(X)$, Fund. Math. 142 (1993) 269–301.

[4] T. Dobrowolski, S.P. Gulko and J. Mogilski, Function spaces homeomorphic to the countable product of $\ell_f^2$, Topology Appl. 34 (1990) 153–160.

[5] T. Dobrowolski, W. Marciszewski and J. Mogilski, Topological classification of function spaces $C_p(X)$ of low Borel complexity, Trans. Amer. Math. Soc. 328 (1991) 307–324.

[6] D.J. Lutzer and R.A. McCoy, Category in function spaces, Pacific J. Math. 90 (1980) 145–168.

[7] D. Lutzer, J. van Mill and R. Pol, Descriptive complexity of function spaces, Trans. Amer. Math. Soc. 291 (1985) 121–128.

[8] W. Marciszewski, On analytic and coanalytic function spaces $C_p(X)$, Topology Appl. 50 (1993) 241–248.

[9] J. van Mill, Infinite-Dimensional Topology. Prerequisites and Introduction (North-Holland, Amsterdam, 1989).

[10] J. Oxtoby, Cartesian products of Baire spaces, Fund. Math. 49 (1961) 157–166.